Linear Algebra I

27/02/2014, Thursday, 18:30-21:30

You are **NOT** allowed to use any type of calculators.

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1 (12+3=15 pts) **Linear equations**

Consider the linear equation

$$\begin{bmatrix} 1 & 3 & 5 & -2 \\ 1 & 4 & 6 & -2 \\ -1 & -1 & -3 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} a \\ b \\ b-2a \end{bmatrix}$$

where a and b are real numbers.

(a) Find all values of a and b for which the equation is consistent. For these values find the general solution of the equation.

(b) Find all values of a and b for which the equation has a unique solution.

2 (7+8=15 pts) **Partitioned matrices and matrix inverse**

Let

$$M = \begin{bmatrix} A & B \\ C & I \end{bmatrix}$$

where all four blocks are $n \times n$ matrices.

(a) Show that M is nonsingular if and only if $A - BC$ is nonsingular.

(b) Let $D = A - BC$. Suppose that D is nonsingular. Show that

$$M^{-1} = \begin{bmatrix} D^{-1} & -D^{-1}B \\ -CD^{-1} & I + CD^{-1}B \end{bmatrix}.$$

3 (5+5+5=15 pts) **Row and column spaces**

Let $A \in \mathbb{R}^{m \times n}$, $B \in \mathbb{R}^{n \times r}$, and $C = AB$. Show that

(a) the column space of C is a subspace of the column space of A.

(b) the row space of C is a subspace of the row space of B.

(c) $\operatorname{rank}(C) \leqslant \min\{\operatorname{rank}(A), \operatorname{rank}(B)\}$.

4 (6+9=15 pts) **Vector spaces**

Consider the vector space of $k \times k$ matrices, i.e. $\mathbb{R}^{k\times k}$.

(a) Let $\lambda \in \mathbb{R}$ and
$$V_\lambda = \{A \in \mathbb{R}^{k\times k} \mid \lambda \text{ is an eigenvalue of } A\}.$$
Show that V_λ is *not* a subspace of $\mathbb{R}^{k\times k}$.

(b) Let $x \in \mathbb{R}^k$ be a nonzero vector and
$$V_x = \{A \in \mathbb{R}^{k\times k} \mid x \text{ is an eigenvector of } A\}.$$
Show that V_x is a subspace of $\mathbb{R}^{k\times k}$.

5 (2+2+4+6+6=20 pts) **Determinants, eigenvalues, and diagonalization**

Consider the matrix
$$M = \begin{bmatrix} 0 & -2 & 1 \\ 1 & 3 & -1 \\ 0 & 0 & \alpha \end{bmatrix}$$
where α is a real number.

(a) Find the determinant of M.

(b) Find all values of α for which M is nonsingular.

(c) Find the eigenvalues of M. [**Hint:** α is an eigenvalue.]

(d) Find all values of α for which M is diagonalizable.

(e) Let $\alpha = 1$. Find a nonsingular matrix T and a diagonal matrix D such that $M = TDT^{-1}$.

6 (10 pts) **Least squares problem**

Let x_i and y_i be real numbers with $i = 1, 2, \ldots, n$. Suppose that
$$x_1 + x_2 + \cdots + x_n = y_1 + y_2 + \cdots + y_n = 0.$$
Show that the least squares solution of
$$\begin{bmatrix} x_1 & 1 \\ x_2 & 1 \\ \vdots & \vdots \\ x_n & 1 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix}$$
is given by
$$a = \frac{x_1y_1 + x_2y_2 + \cdots + x_ny_n}{x_1^2 + x_2^2 + \cdots + x_n^2} \quad \text{and} \quad b = 0.$$

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10 pts free